Lezione 10

$$\omega \in \Omega^k(M) \quad \text{chiusa, cioè } d\omega = 0$$

$$\overset{?}{\Longrightarrow} \exists\, \eta \in \Omega^{k-1}(M) \text{ t.c. } \omega = d\eta$$

Sì se M è contrattile

$$F \in \Omega^2(M) \quad dF = 0 \quad \underset{\text{se } M \text{ contrattile}}{\Longrightarrow} \exists\, A \in \Omega^1(M) \text{ t.c. } F = dA$$

M Lorentziana

loc. $A = f\,dt + g\,dx + h\,dy + l\,dz$

Varietà pseudo-Riemanniane

(M, g) $\quad M$ varietà $\quad g$ tensore metrico $\quad$ segnatura (p, m)

Riemanniana se $m = 0$

Lorentziana se $m = 1$

$\gamma : I \to M$ $\qquad p \in M \quad v \in T_pM \quad \|v\| = \sqrt{|\langle v, v\rangle|}$

$$L(\gamma) = \int_I \|\gamma'(t)\|\, dt$$ LUNGHEZZA di γ

$\varphi : \overset{u}{J} \to \overset{t}{I}$ diffeomorfismo fra intervalli

RIPARAMETRIZZAZIONE $\quad \eta = \gamma \circ \varphi$

<u>Prop</u>: $L(\gamma) = L(\eta)$

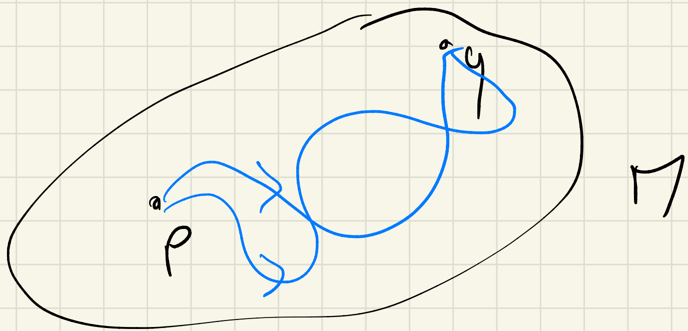
<u>dim</u>: $$\int_I \|\gamma'(t)\|\, dt = \int_J \|\gamma'(\varphi(u))\|\, |\varphi'(u)|\, du$$

CHAIN RULE $d(f \circ g)_p = df_{g(p)} \circ dg_p$

$$\int_{\bar{U}} \| \eta'(u) \| \, du$$

Se (M, g) è Riemanniana, M connessa si può definire una distanza

$p, q \in M$ $\quad d(p,q) = \inf \left\{ L(\gamma) : \gamma : [a,b] \to M, \ \gamma(a) = p, \ \gamma(b) = q \right\}$



Prop: (M, d) è uno SPAZIO METRICO

cioè $\quad$ 1) $d(p,q) = 0 \iff p = q$

$\Leftarrow$ ovvia

2) $d(p,q) = d(q,p)$

3) $d(p,q) \leq d(p,r) + d(r,q)$

Es: $\mathbb{R}^n$, $d(x,y) = \|x-y\| =$

$$= \sum_{i=1}^{n} \sqrt{x_i^2 - y_i^2}$$

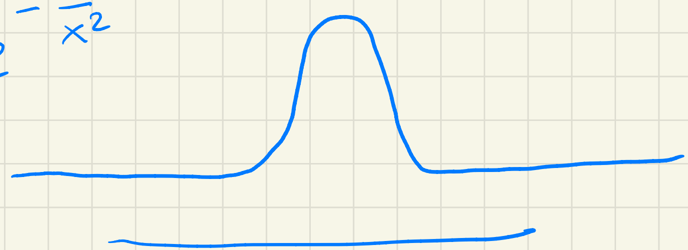
dim:

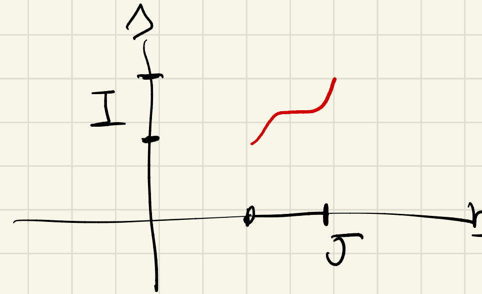
$\gamma : I \longrightarrow M$

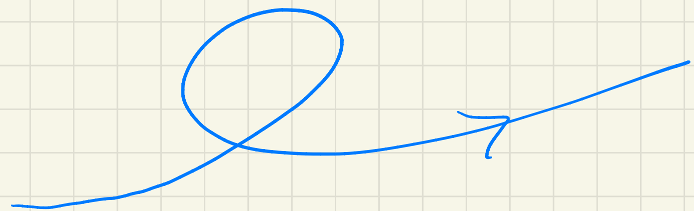
Def: Se $\varphi : J \longrightarrow I$ è monotona (cioè $\varphi' \geq 0$ oppure $\varphi' \leq 0$) e suriettiva

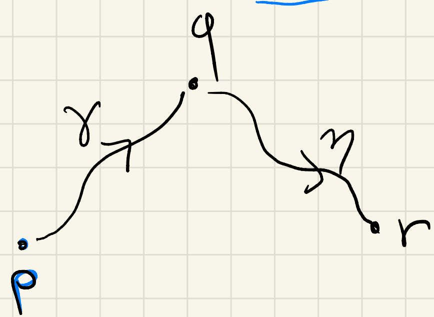
la chiamo comunque riparametrizzazione

$e^{-\frac{1}{x^2}}$









concateno γ e η

$\gamma * \eta$

$$L(\gamma * \eta) = L(\gamma) + L(\eta)$$

$\exists\ \gamma_i$ da p a q $\quad L(\gamma_i) \to d(p,q)$

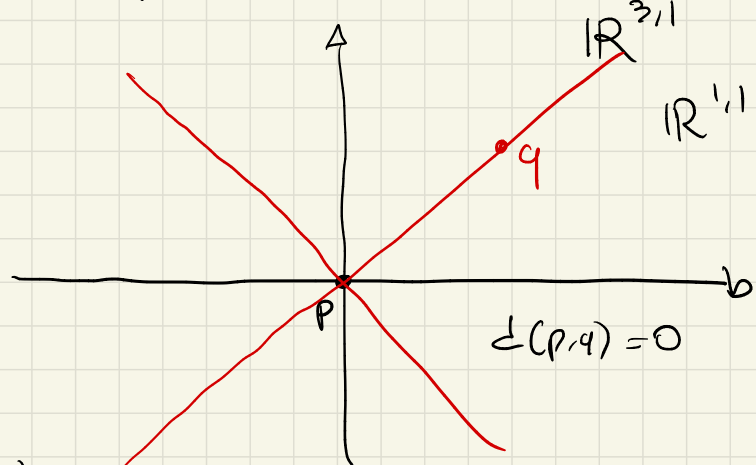
η_i q a r $\quad L(\eta_i) \to d(q,r)$

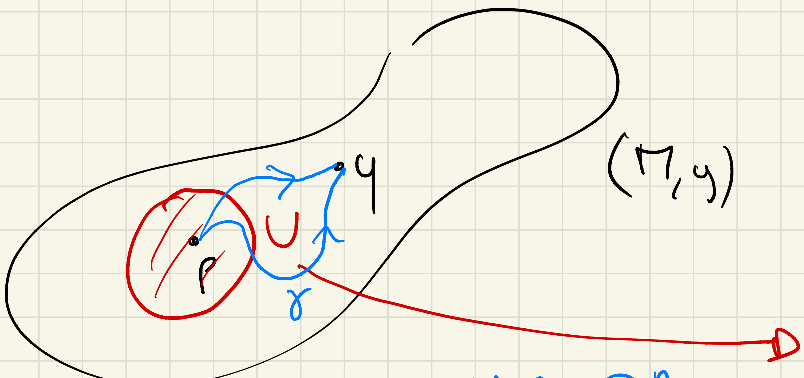
$$L(\gamma_i * \eta_i) = L(\gamma_i) + L(\eta_i) \to d(p,q) + d(q,r)$$

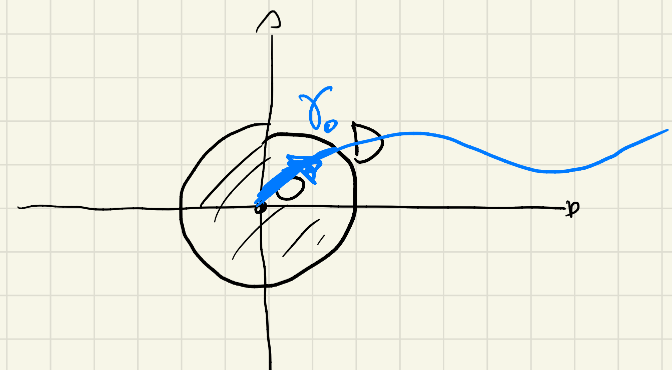
$$d(p,r) := \inf_{\alpha \text{ da } p \text{ a } r} L(\alpha) \leq d(p,q) + d(q,r)$$

Devo mostrare che $d(p,q) > 0$

se $p \neq q$.







$\varphi : U \xrightarrow{\sim} \mathbb{R}^n$

carta

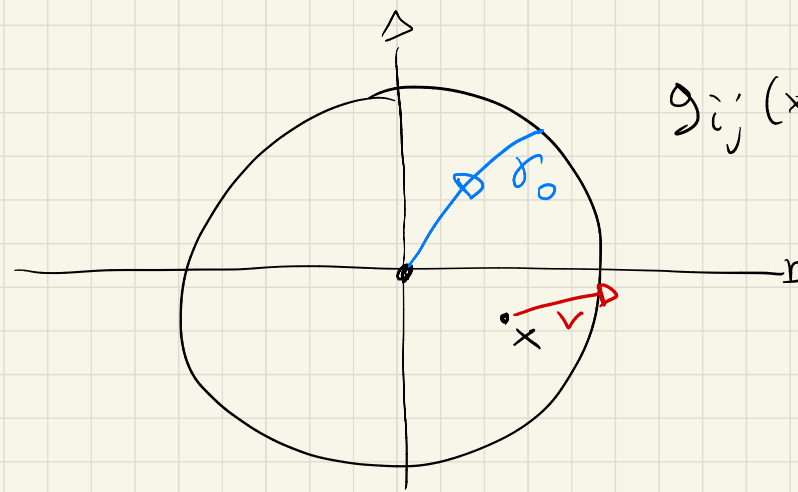
$p \mapsto 0$

$D = \{x \mid x_1^2 + \dots + x_n^2 \le 1\}$

g_{ij} simm. def + che dipende da $x \in \mathbb{R}^n$

Tesi: $L(\gamma) > M > 0$ con M che non dipende da γ

Tesi: $L(\gamma_0) > M > 0$



$g_{ij}(x)$

$g_{ij}(x)$ $\quad\quad$ $\delta_{ij}(x)$

$v \in T_x\mathbb{R}^n = \mathbb{R}^n$

$k_x \|v\|^g \le \|v\|^E \le K_x \|v\|^g$ $\quad\quad$ $k > 0$ non dipende da v

$k > 0$

Lemma: Due norme su uno V di dim finita sono sempre equivalenti

$$k \|v\|^g \leq \|v\|^E \leq K \|v\|^g \qquad k = \min_{x \in D} k_x \qquad K = \max_{x \in D} K_x$$

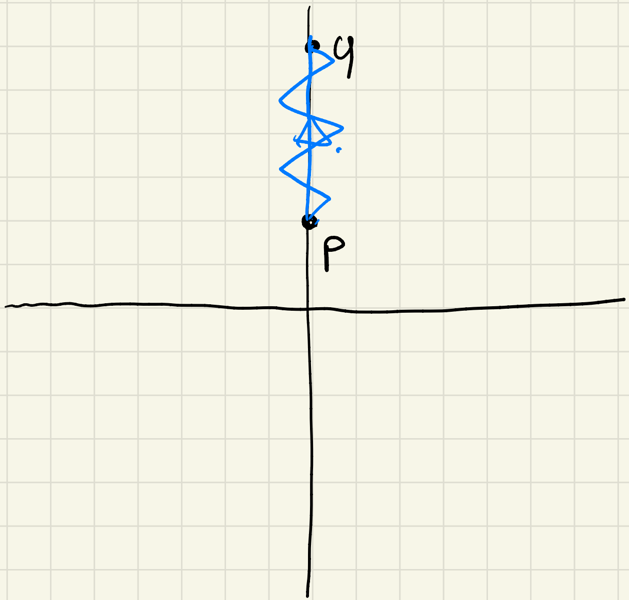
Cor: $\|v\|^g \geq \frac{\|v\|^E}{K}$

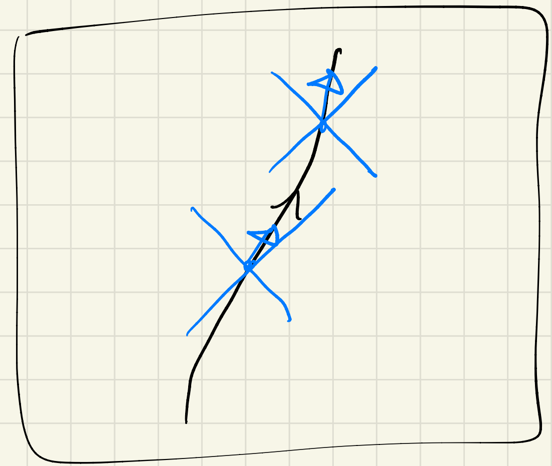
Cor: $L^g(\gamma_0) = \int \|\gamma_0'(t)\|^g \, dt \geq \frac{1}{K} \int \|\gamma_0'(t)\|^E dt$

$$= \frac{1}{K} \underset{\substack{\vee\!\shortparallel \\ 1}}{L^E(\gamma_0)} \geq \frac{1}{K} \qquad M = \frac{1}{K}$$

Def (M,g) Lorentziana γ di tipo tempo

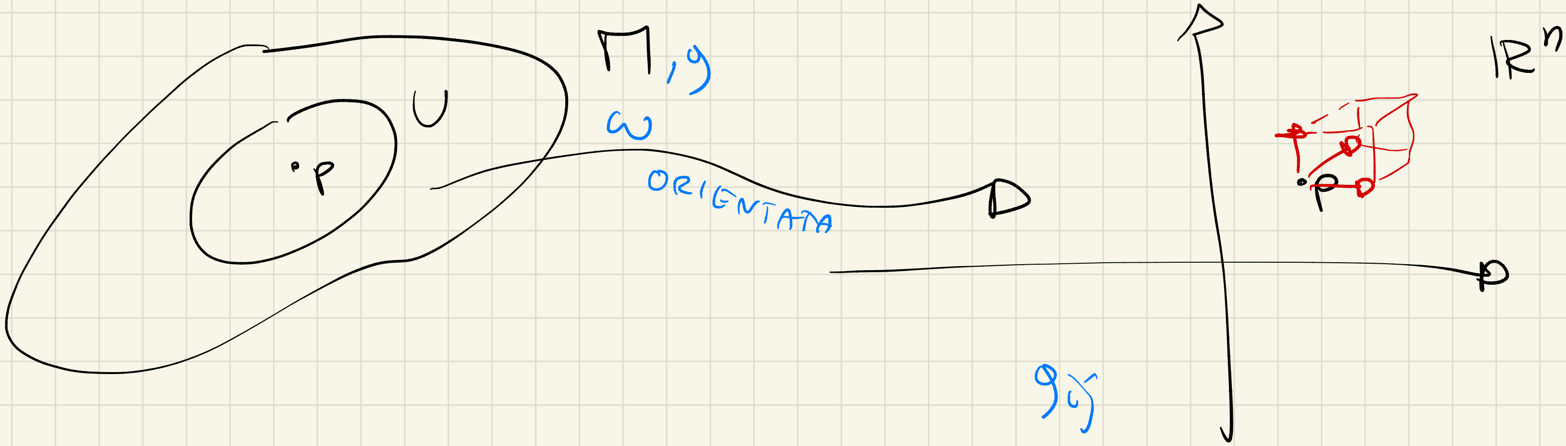
$L(\gamma)$ è il TEMPO PROPRIO





(M,g) pseudo-Riemanniana orientata $\dashrightarrow$ $\omega \in \Omega^n(M)$ forma volume

In carte:



$$\omega = f\, dx^1 \wedge \_\_ \wedge dx^n$$

$$f > 0$$

Prop:

$$f = \sqrt{|\det g_{ij}|}$$

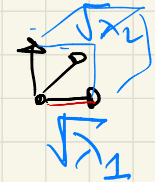
$$\omega = \sqrt{|\det g_{ij}|}\, dx^1 \wedge \_\_\_ \wedge dx^n$$

dim: Fisso $p \in \mathbb{R}^n$ Teo spettrale $g_{ij}(p)$

posso prendere una base ortonormale e ottenere g_{ij} diagonale

$$g_{ij} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_i > 0$ per semplicità



$\square$

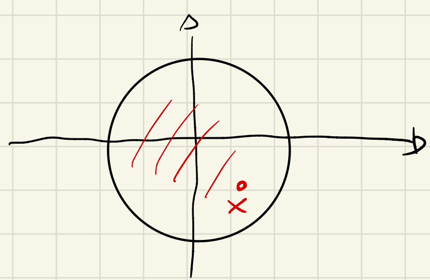
SPAZIO IPERBOLICO

$(\mathbb{R}^n, g^E)$ $\quad K \equiv 0$

MODELLO DEL DISCO: $\quad M = B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$

(M, g)

$n = 2$

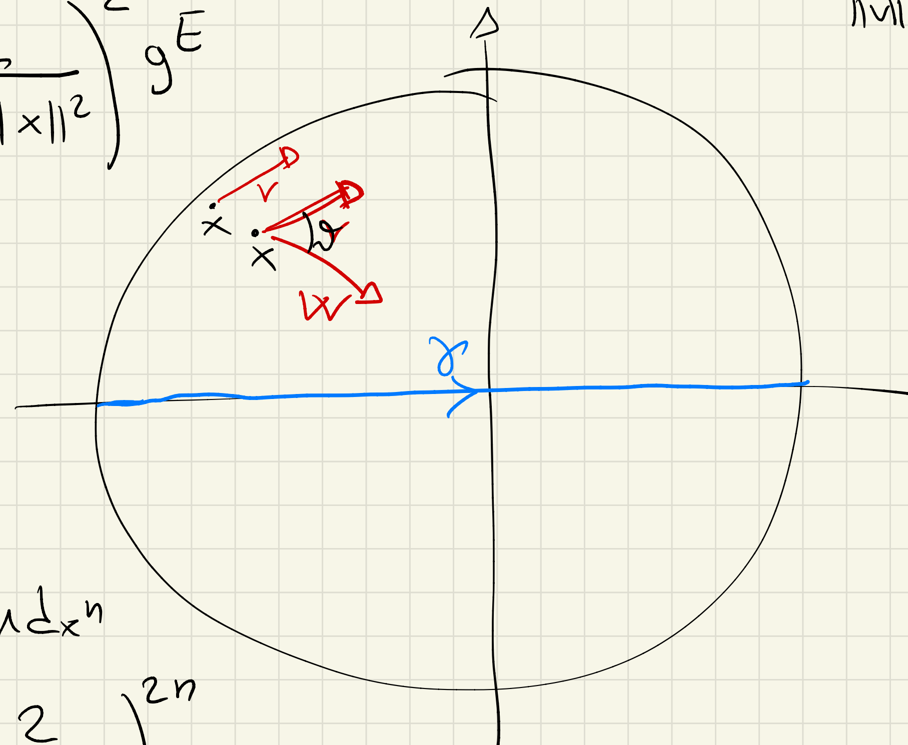


$$\lambda : B^n \to (0, +\infty)$$

$$g(x) = \lambda(x) \cdot g^E \qquad \lambda(x) > 0$$

$$\cos\vartheta = \frac{\langle v, w\rangle}{\|v\| \|w\|}$$

$$g^H(x) = \left(\frac{2}{1-\|x\|^2}\right)^2 g^E$$



$$\|v\|^H = \frac{2}{1-\|x\|^2} \|v\|^E$$

<u>Ex</u>: $L(\gamma) = +\infty$

$$\omega^H = \sqrt{\det g_{ij}^H}\, dx^1 \wedge \dots \wedge dx^n$$

$$\det g^H = \left(\frac{2}{1-\|x\|^2}\right)^{2n}$$

$$\omega^H = \left(\frac{2}{1-\|x\|}\right)^n dx^1 \wedge \dots \wedge dx^n$$

CONFORME = STESSI ANGOLI